Resolving the Banach-Tarski Paradox

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Chapter 1

Introduction

1.1 Measure Theory and Lebesgue Measure

Measure theory is a field of mathematics dedicated to the properties of *measures*: functions that assign a nonnegative number to a set. The main motivation in measure theory is to mathematically describe the notion of the n-dimensional volume of a set in $\mathbb{R}^n$: in particular, to determine the length, area, or volume of a set in $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$ respectively. For many sets with nice properties, this theory was developed by mathematicians several millennia ago, where civilisations such as the Greeks were able to determine the area of regular shapes and solids, such as polygons, polyhedra, and balls. But for less regular sets, the problem of measure remained.

The need for a well-defined measure on $\mathbb{R}^n$ became a major issue in the 19th century through the development of integration, and, in particular, Riemann integration. The conditions for a function to be Riemann integrable is related to the size of the set of points at which the function is discontinuous. However, it was unknown how to define such a measure to ensure that if the size of the set was sufficiently small, then the function would be Riemann integrable.

This was resolved with the advent of Lebesgue measure, introduced by the French mathematician in 1902. Not only did his notion of a measure accurately show when a function is Riemann integrable, he was able to weaken the conditions of integrability by creating the more generalised notion of the Lebesgue integral. Moreover, his definition of a measure extended past $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ to any dimension, while still retaining consistency by agreeing with the natural notions of length, area, and volume.

Definition 1.1 (Lebesgue's Definition of Measure). Let E be a subset of $\mathbb{R}^n$. The *Lebesgue (outer) measure* of E, denoted $\mu(E)$, is defined to be

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(R_i) : E \subset \bigcup_{i=1}^{\infty} R_i \right\},$$

where each $R_i = [a_1^i, b_1^i] \times \cdots \times [a_n^i, b_n^i]$ is a closed (possibly empty) rectangle, and the measure of each R_i is given by $\mu(R_i) = (b_1^i - a_1^i) \times \cdots \times (b_n^i - a_n^i)$.

So Lebesgue measure involves approximating a set by covering it with rectangles, and then adding together the volumes of the rectangles. This definition of a measure was highly lauded by the mathematical community at the time for elegantly unifying integration theory and measure theory. Moreover, it appeared that Lebesgue measure was consistent with previous theorems of mathematics in measure theory and accurately modelled the notion of volume in the real world.

Lebesgue measure unlocked an area of mathematics that had previously been plagued with inconsistencies. With Lebesgue's definition of a measure on $\mathbb{R}^n$, it was possible consider measures on more general spaces, leading to the current generalised definition of a measure.

Definition 1.2. A *measure* is a function on subsets $\mathcal{M}$ of a set X, where $\mathcal{M}$ is a σ-algebra and so is closed under complements and countable unions and intersections, as well as containing $\emptyset$ and X itself. The measure $\mu : \mathcal{M} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$, and that if $E_1, E_2, \ldots$ is a countable family of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

Thus a measure is a function that assigns each set in $\mathcal{M}$ a nonnegative real number, and satisfies that the measure of the union of disjoint sets is equal to the sum of the measure of each set. The latter property is known as countable additivity.

1.2 Non-Measurable Sets and the Banach-Tarski Paradox

However, it was discovered through an example of Giuseppe Vitali in 1905 that not every set had a well-defined measure. For one would expect every set to be as "big on the inside as the outside", which means that they would satisfy Carathéodory's criterion for measurability.

Definition 1.3 (Carathéodory's Criterion). A set $E \subset \mathbb{R}^n$ is *measurable* if for all sets $A \subset \mathbb{R}^n$,

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E).$$

Vitali gave an example of a set in $\mathbb{R}^n$ that did not satisfy this criterion, and so was non-measurable. With this, many aspects of Lebesgue's theory of measure would fail, as this theory relied on sets being measurable. In particular, Lebesgue measure is not countably additive on all of $\mathbb{R}^n$. This was seen most strongly through the Banach-Tarski paradox, which built upon the earlier paradox created by Felix Hausdorff.

Theorem 1.4 (**The Hausdorff Paradox**). *One can decompose S^2, the unit sphere in $\mathbb{R}^3$, into four pieces A_1, A_2, A_3, and C, such that C is countable (and hence has measure zero), and such that A_1, A_2, A_3, and $A_2 \cup A_3$ are each congruent to each other.*

As Lebesgue measure has the property that congruent sets have the same volume, this suggests that there are subsets of S^2 that are simultaneously one-third and one-half of the sphere; hence the paradox. The contradiction with intuition of Lebesgue non-measurable sets is shown most strongly through the paradox of Stefan Banach and Alfred Tarski, which uses the Hausdorff paradox to create a paradoxical decomposition of the unit ball in $\mathbb{R}^3$.

Theorem 1.5 (**The Banach-Tarski Paradox**). *One can decompose B^3, the unit ball in $\mathbb{R}^3$, into five disjoint pieces, and by rotating and translating them, these five pieces can be recombined in order to create two balls of the same size as the original ball.*

This highly contradictory result became a very contentious issue in mathematics following its publication in 1924. For as Lebesgue measure supposedly models volume in real life accurately, it seems to suggest that it is possible to create new volume out of nothing. Of course, this is not the case: the five pieces used in decomposing B^3 are non-measurable.

1.3 The Problem of Measure

The Banach-Tarski paradox is a proof that one cannot use Lebesgue measure to model volume on *every* subset of $\mathbb{R}^n$, as some subsets may be non-measurable. This answered an earlier question of Lebesgue's, which is the main question of this paper. We present this question in a slightly more modern formulation:

Main Question of the Paper (**The Problem of Measure**). *Using the standard axioms of set theory, does there exist a measure on all the subsets of $\mathbb{R}^n$ that is countably additive, isometry invariant, and gives the unit cube volume 1? If so, is this measure unique?*

The motivation for this question is, of course, the notion of length, area, and volume in real life. When it was proved that Lebesgue measure is the unique measure satisfying these properties restricted to (Lebesgue) measurable sets, it was expected that Lebesgue measure would be the solution to this problem. Thus it came as great surprise to the mathematical community that no such measure exists, as Vitali's construction of a non-measurable set and the Hausdorff and Banach-Tarski paradoxes show. The result of this is the disturbing disparity between real life and mathematics; using the standard techniques of mathematics, it is not possible to accurately model things such as real-life volume.

The question that we examine in this paper is aimed at resolving this disparity: we ask that if we weaken certain conditions of the problem of measure, can we answer it in the positive? Our motivation behind this is that by answering the problem of measure in the positive, we will be in a sense resolving the Banach-Tarski paradox.

There are, of course, plenty of conditions to examine: in the original formulation, we have demanded that we use the standard definitions of set theory, that the measure we work with is countably additive, that the measure is invariant under isometries (that is, under rotations, translations, and reflections), and that the unit cube is given volume 1. In this paper, we look at what are perhaps the three most common ways of resolving the problem of measure:

(1) We consider non-additive measures μ on $\mathbb{R}^n$, such that there exist $E_1, E_2 \subset \mathbb{R}^n$ with $\mu(E_1 \cup E_2) \neq \mu(E_1) + \mu(E_2)$. This allows us, for example, to extend Lebesgue measure to every subset of $\mathbb{R}^n$, including previously non-measurable sets. In order to do so, however, we must change the definition of a measure.

(2) We consider measures using different axioms of set theory. The standard axioms of set theory in mathematics are the Zermelo-Fränkel axioms together with the Axiom of Choice, which we denote ZF+AC. The construction of non-measurable sets, and in particular the Banach-Tarski paradox, all rely on the Axiom of Choice, which leads to highly pathological, seemingly unrealistic sets. By working in a system of mathematics without this axiom, for example, it is consistent that the problem of measure is resolved in the positive.

(3) We consider restricting measures to certain "measurable" sets. This is the standard method of resolving paradoxes in measure theory, and is used in the modern formulation of the construction of Lebesgue measure. The σ-algebra $\mathcal{M}$ for Lebesgue measure is the subsets of $\mathbb{R}^n$ that

satisfy Caratheodory's criterion. With this definition of a measure, the problem of measure is resolved in the positive.

We could also consider weakening other criteria, such as considering isometry variant measures, measures that give the unit cube zero or infinite volume, or measures on sets other than $\mathbb{R}^n$. In general, however, by weakening these criteria we lose our main application of the problem. Isometry variance implies that the volume of a set may change merely by rotating or translating it, which is in direct contradiction of what we know of in real life; the Cosmological Principle states that at large scales, the Universe is isotropic and homogeneous, whereas an isometry variant notion of volume would contradict this.

Similarly, it would seem ludicrous to consider a measure the unit cube as having zero or infinite volume, as if we nevertheless require that such a measure be countably additive and isometry invariant, then the vast majority of non-trivial sets would also have either zero or infinite volume. This approach is taken somewhat in the definition of Hausdorff measure, a measure used to find the (possibly non-integer) Hausdorff dimension of a set. But Hausdorff measure is not very informative outside showing the dimension of a set; its use is primarily in determining the dimension of fractal sets, not modelling n-dimensional volume.

Finally, we could consider models of space other than $\mathbb{R}^n$, the standard model. Several academics from the realms of mathematics, physics, and philosophy have suggested countable models of space as opposed to the uncountable reals, such as countable models of the computable reals. These have the possibility of resolving the problem of measure by redefining the notion of space completely. However, examining such models and their possible resolution of the problem are beyond the scope of this paper.

Chapter 2

Additivity

The unexpected result of the duplication of a set in the Banach-Tarski paradox occurs because we expect a measure to be additive: that the volume of the union of sets is the same as the sum of the volume of each set. Additivity is, in fact, part of the definition of measure. But if we remove this necessity, then is it possible to resolve the problem of measure?

Question 1. *Using the standard axioms of set theory, does there exist a non-additive measure on all the subsets of* $\mathbb{R}^n$ *that is isometry invariant, and gives the unit cube volume 1? If so, is this measure unique?*

2.1 Finitely Additive Measures

Vitali's construction of a non-measurable set in $\mathbb{R}^1$ shows that the problem of measure (with countable additivity) is answered in the negative. Similarly, the Banach-Tarski paradox shows that even with finite additivity (as opposed to countable additivity), the problem of measure is answered in the negative for $\mathbb{R}^3$. In fact, the Banach-Tarski paradox implies this for $\mathbb{R}^n$ for all $n \geq 3$.

On the other hand, the Banach-Tarski paradox has nothing to say about the possibility of a finitely additive measure acting on all subsets of $\mathbb{R}^1$ or $\mathbb{R}^2$, as the paradox belongs in a higher dimension. Moreover, the non-measurability of the Vitali set requires that Lebesgue measure be countably additive. So is it possible to extend Lebesgue measure on $\mathbb{R}^1$ and $\mathbb{R}^2$ such that the extended measure is only finitely additive?

It turns out that this is, in fact, the case: there does exist a finitely additive extension of the Lebesgue measure onto all subsets of $\mathbb{R}^1$ and $\mathbb{R}^2$ such that the measure is isometry invariant. Thus this slight weakening of the problem of measure results in a positive solution. It must be noted, however, that the standard proof of this result requires the use of the Axiom of Choice; it is a theorem of ZF+AC. In weaker forms of Zermelo-Fränkel set theory, this result cannot be proved.

2.2 Non-Additive Measures

While the resolution of the problem of finitely additive measures in $\mathbb{R}^1$ and $\mathbb{R}^2$ is reassuring, we know due to the Banach-Tarski paradox that no such resolution exists in $\mathbb{R}^3$. This is, of course, the dimension in which we are most interested, being our standard model for space. So we naturally wonder whether we can resolve the problem of measure using non-additive measures.

The standard method in this approach is to consider outer measures.

Definition 2.1. An *outer measure* is a function on all subsets of a set X. The outer measure $\mu_* : \mathcal{P}(X) \to [0, \infty]$ satisfies $\mu_*(\emptyset) = 0$, that if $E_1 \subset E_2$ then $\mu_*(E_1) \leq \mu_*(E_2)$, and that if $E_1, E_2, \ldots$ is a countable family of sets in $\mathcal{M}$ then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

The last property in the definition above is known as countable subadditivity. An obvious example of an outer measure is our original definition of Lebesgue measure: we can apply this to every subset of $\mathbb{R}^n$, and all of the conditions of the definition above are satisfied. It is only when we specify that Lebesgue measure must be countably additive, as opposed to countably subadditive, that we must restrict the measure to the σ-algebra of (Lebesgue) measurable sets.

In fact, we can consider forms of non-additive measures even stronger than outer measures; we can, for example, consider metric outer measures.

Definition 2.2. A *metric outer measure* is an outer measure satisfying $\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$ for all A, B with $\inf\{|x - y| : x \in A,\ y \in B\} > 0$.

So a metric outer measure is additive on sets of positive distance apart. Again, Lebesgue outer measure is a metric outer measure on $\mathbb{R}^n$. Of course, it must be noted that Lebesgue outer measure is not the only outer measure acting on $\mathbb{R}^n$ that is isometry invariant and normalises the unit cube. In 1884, nearly twenty years before Lebesgue's theory of measure was introduced to the world, Georg Cantor proposed an outer measure for $\mathbb{R}^n$.

Definition 2.3 (Cantor's Outer Measure). Let $x \in \mathbb{R}^n$, $r > 0$, and define $\bar{B}_r(x) = \{y \in \mathbb{R}^n : |y - x| \leq r\}$. Then if A is a bounded subset of $\mathbb{R}^n$, we define $A_r = \bigcup_{x \in A} \bar{B}_r(x)$ for all $r > 0$. We define $\mu(A_r)$ via Riemann integration, and then define the Cantor measure of A to be $\mu(A) = \lim_{r \to 0} \mu(A_r)$.

There are clearly several issues with this definition: for one, Cantor does not clearly show how to calculate $\mu(A_r)$, other than relate it to Riemann integration. Unlike Lebesgue measure, it does not distinguish between a set and its closure, and is therefore not additive. For example, we have that

$$1 = \mu([0,1]) = \mu([0,1] \cap \mathbb{Q}) = \mu([0,1] \setminus \mathbb{Q}).$$

This alone is enough to resolve the Banach-Tarski paradox in some sense: the pieces used in the decomposition of the ball are each dense in the ball, and so the measure of each piece would be equal to that of the ball itself.

It might seem strange in light of modern measure theory that Cantor would suggest a non-additive measure. In the development of measure theory in the late 19th century, however, additivity was not seen as important a part of measure theory as it is today. Most early versions of measure were not necessarily additive, until Emile Borel in 1898, who defined his measure to be countably additive by restricting his measure only to certain "Borel measurable sets", the smallest σ-algebra containing the open sets (and hence the smallest containing the closed sets).

This approach, of the necessity of additivity of a measure, was in fact criticised at the time. Arthur Schoenflies, compiling a two-volume report on the theory of sets and transfinite numbers at the turn of the 20th century, was critical of Borel's definition of measure, suggesting that unlike other theories of measure in use and development at the time, Borel's measure was of little further use in applications of measure theory at the time. Furthermore, Schoenflies disagreed with Borel's definition of measure being countably additive, as though it held for Borel's measure, other popular measures in use at the time did not necessarily have this property. In effect, Schoenflies accused Borel, by defining his own measure to be countably additive, of forcing all measures to share this property, though many had not previously done so; Schoenflies saw no need for a measure to be additive.

It must be noted, however, that Schoenflies glossed over some of the many fallbacks of early definitions of a measure. The primary application of measure theory in the 19th century was the development of Riemann integration. But many measures failed to sufficiently bridge these two areas of mathematics; for example, with some measures the area under the graph of a function is not equal to the Riemann integral of the function. Similarly, one of the main motivations was to determine the conditions for a function to be Riemann integrable. Lebesgue discovered that the key condition involves the set of discontinuities of a function having (Lebesgue) measure zero, but many earlier forms of measure gave positive measures to sets of Lebesgue measure zero, and gave measure zero to nowhere dense sets, despite these possibly having positive Lebesgue measure.

So why do we demand today that a measure be additive? It is primarily due to the power of Lebesgue measure. Henri Lebesgue was motivated by Borel's work on measures and continued that in the development of his own version of measure theory. This measure greatly surpassed all other measures before it in its generality and consistency, solving many of the problems in integration theory, the main application of measure theory. On the other hand, there remained a necessity to restrict Lebesgue measure to measurable sets in order to have countable additivity.

However, we have seen that Lebesgue outer measure is, in fact, enough in some sense to resolve the Banach-Tarski paradox, by merely noting that as the pieces used in the decomposition of B^3 are not of positive distance apart from each other, their measures are not additive. Moreover, we can restrict Lebesgue outer measure to the σ-algebra of Lebesgue-measurable sets in order to obtain a countably additive measure. Unfortunately, we are begging the question here: we have retreated to defining sets as non-measurable in order to regain the nice properties of a measure, as opposed to the less useful properties of an outer measure. An outer measure, for example, does not guarantee that $\int_{A \cup B} f = \int_A f + \int_B f$ for disjoint sets A, B unless A and B are of positive distance apart, despite the intuitive nature of this under Riemann integration. We must first consider whether A and B are Lebesgue-measurable in order to prove that this statement holds, in which case we might as well have worked with Lebesgue measure all along.

It must be noted, however, that the development of integration theory over the last century has nearly all been based on the foundations of measure theory created by Lebesgue, where countable additivity is seen as a necessity and measures are only defined over σ-algebras. Though this has led to a seemingly rich theory of measure and integration, it is difficult to say whether similarly powerful theories could have been created had other definitions of measure been prevalent in this time.

2.3 Verdict

Based on the current development of measure and integration theory, we must conclude that using non-additive measures removes too much of these areas, though perhaps because the development has had additive measures in mind. Metric outer measures may be similar enough to measures and still explain the paradoxical decompositions in the Banach-Tarski paradox, but one must restrict this outer measure to certain sets in order to obtain the more powerful and useful theorems of measure and integration theory. Moreover, non-additive measures remove the real-life intuition of the conservation of volume despite decompositions and violate physical laws; in fact, if volume of a solid body is not conserved under decompositions then clearly the law of conservation of energy is contradicted. We cannot have the existence of sets $E_1, E_2 \subset \mathbb{R}^n$ with $\mu(E_1 \cup E_2) \neq \mu(E_1) + \mu(E_2)$, because then we are either losing mass or gaining mass unexpectedly, or equivalently losing or gaining energy without justification. Thus it seems infeasible to consider non-additive measures as a satisfactory resolution of the problem of measure in $\mathbb{R}^n$, where $n \geq 3$, though as we noted earlier, the problem is solved in the positive by a finitely-additive extension of Lebesgue measure in $\mathbb{R}^1$ and $\mathbb{R}^2$.

Chapter 3

Axioms

The problem of measure has no satisfactory resolution when it is assumed that the standard axioms of set theory are used. This is not surprising; mathematics relies on the consistency of using the same axioms over all fields of mathematics, and so it is logical to base the problem on the structure of set theory used in all mathematics. But as the problem of measure is intrinsically related to its application of modelling the notion of volume in space, is it possible that the resolution of the problem might be in the consideration of a different structure of set theory to model mathematics, and hence to model space?

Question 2. *Under what axioms of set theory does there exist a measure on all the subsets of* $\mathbb{R}^n$ *that is isometry invariant, and gives the unit cube volume 1? Is this measure unique?*

3.1 The Zermelo-Fränkel Axioms of Set Theory

Lebesgue's theory was developed at the turn of the 20th century. At the same time, the foundations of mathematics itself were being rebuilt via the development of axiomatic set theory. Mathematics had traditionally been built upon the backbone of Euclidean geometry; geometry was considered the foundation upon which all mathematics could be built. This was seen to be fallacious in the development of non-Euclidean form of geometry by Nikolai Lobachevsky in 1826. This new form of geometry, named hyperbolic geometry, was proven to be consistent if and only if Euclidean geometry were consistent.

Without one pure form of geometry to base the formulation of mathematics upon, many mathematicians began to consider using axiom systems as a basis for set theory, which, in conjunction with first-order logic, would form a foundational system of mathematics. Early attempts of axiom systems by Georg Cantor and Richard Dedekind showed the many pitfalls involved in set theory; though these systems seemed to be a simple model for set theory, paradoxes such as Russell's paradox indicated the lack of consistency of such a system.

The current standard axiom system used as the foundation for nearly all areas of modern mathematics was first proposed by Ernst Zermelo in 1908 and developed further by Abraham Fränkel in 1922. The current version of Zermelo-Fränkel, or ZF, set theory consists of eight axioms on the existence, construction, structure, and relation of sets. With these axioms, one further axiom is usually added to create the standard set theoretic model in mathematics: the Axiom of Choice.

Axiom 3.1 (The Axiom of Choice). Given any family $\mathcal{F}$ of nonempty sets, there exists a function f that assigns to each member A of $\mathcal{F}$ an element $f(A)$ of A.

The Axiom of Choice seems quite unassuming; it merely states that if we have a group of sets, then we can define a function that chooses an element of each set. In fact, the Axiom of Choice can be proved from the other eight axioms if the family of sets is finite. In some cases, if the family of sets is countable, then again this statement is merely a theorem based on the other eight axioms. But for arbitrary families of sets, the Axiom of Choice is independent of the other eight axioms.

Though ZF+AC is the standard axiomatic model of set theory used in modern mathematics, it is certainly not the only model. For the Zermelo-Fränkel axioms of set theory have been criticised for being either too strong, as many theorems of mathematics require much weaker axioms than those presented, or too weak, as mathematical problems such as the Continuum Hypothesis are unprovable in this axiomatic system. Moreover, Gödel's Incompleteness Theorems show that it is impossible to prove the consistency of ZF+AC using ZF+AC itself.

The main criticism, however, of the Zermelo-Fränkel Axioms of set theory are purely on the Axiom of Choice. Critics point out that it is nonconstructive: it states the existence of a choice function, but does not tell how the choice is made, leading to the famous quotation:

> The Axiom of Choice is necessary to select a set from an infinite number of socks, but not an infinite number of shoes.
>
> – *Bertrand Russell*

That is, it is possible to define a function to select from an infinite number of pairs of shoes by choosing the left shoe, but without the Axiom of Choice, one cannot assert that such a function exists for pairs of socks, because left and right socks are identical. While Russell's statement puts the criticism of the nonconstructivism in a humourous light, the fact that the Axiom of Choice does not specifically describe how to create, for example, a non-measurable set is a contentious issue.

3.2 The Axiom of Choice and the Banach-Tarski Paradox

The Axiom of Choice is often targeted as the chief source of blame for the Banach-Tarski paradox, because it is required in order to construct the pieces used in the paradoxical decomposition of B^3. In fact, all non-measurable sets require the Axiom of Choice; the existence of a non-measurable set is a theorem of ZF+AC. If the underlying foundations of mathematics are merely ZF without the Axiom of Choice, then the existence of a non-measurable set is unprovable, and so it is consistent to assume that every set is measurable. In fact, the strongest set theoretic structure we can use that the existence of a non-measurable set is unprovable is ZF+DC, where DC is the Axiom of Dependent Choice.

Axiom 3.2 (The Axiom of Dependent Choice). Given a nonempty set X and a binary relation R on X, there exists a sequence (x_n) in X such that $x_n R x_{n+1}$ for each $n \in \mathbb{N}$.

The Axiom of Dependent Choice implies the Axiom of Choice for countable families of sets, but the Axiom of Choice for arbitrary families of sets implies, and is strictly stronger than, the Axiom of Dependent Choice. Though it is weaker, the Axiom of Dependent Choice is sufficient for the development of most areas of mathematics. We can therefore consider the set theoretic system ZF+DC+GM, where GM is an axiom that states that there exists a countably additive, isometry invariant measure on all subsets of $\mathbb{R}^n$ that normalises the unit cube. It has been proved that this axiomatic structure for mathematics is consistent if and only if ZF is consistent.

So non-measurable sets are non-constructive, as they require the Axiom of Choice, and are seemingly irrelevant in physical world applications, as without a constructive choice we cannot physically construct the non-measurable sets in the Banach-Tarski paradox in order to duplicate a ball. Nevertheless, the usage of AC leads to physical contradictions. Thus a natural resolution of this paradox would be to just not use the Axiom of Choice when considering the problem of measure, and instead work with the axiomatic system ZF+DC+GM. With this, we have, in effect, solved the problem of measure: under these axioms of set theory, Lebesgue measure satisfies the properties of the problem, and is the unique measure with this property as it is the unique measure over Lebesgue-measurable sets.

Despite this simple solution, this is generally not the method used to resolve the problem of measure. The main issue with such a resolution is that the Axiom of Choice is considered necessary for many other areas of mathematics, such as in functional analysis, where it is used to prove the Hahn-Banach theorem.

Theorem 3.3 (The Hahn-Banach Theorem). *Let V be a vector space, and p be a sublinear functional on V such that $p : V \to \mathbb{C}$ satisfies $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$ for all $x, y \in V$, $\alpha \in [0, \infty)$. Then if f is a linear functional on a subspace M of V satisfying $f(x) \leq p(x)$ for all $x \in M$, there exists a linear functional F on the whole of V such that $F(x) = f(x)$ for all $x \in M$ and $F(x) \leq p(x)$ for all $x \in V$.*

This theorem, though proved using the Axiom of Choice, is strictly weaker than the axiom itself; one cannot prove that AC holds using ZF+HB, where HB is the axiom that the Hahn-Banach theorem holds for all vector spaces V and sublinear functionals p. Moreover, assuming ZF+HB as a system of axioms for set theory, it possible to construct a Lebesgue non-measurable set. So if we wish to assume that every set is Lebesgue-measurable, then we cannot prove the Hahn-Banach theorem.

Unfortunately, it proves too costly to do this. The Hahn-Banach theorem is one of the cornerstones of functional analysis, a key area of study in modern mathematics. To remove this theorem would vastly diminish the possible study in this area and reduce our mathematical knowledge greatly. Moreover, it is not possible to have one system of axioms for one area of mathematics and another system of axioms for another area of mathematics, due to the possible overlap of these areas. In this case, we cannot suggest having ZF+DC as our axiomatic system for measure theory and ZF+HB as our axiomatic system for functional analysis, as the extremely close relationship between measure theory and functional analysis would lead to inconsistencies. The threat of such inconsistencies is too dangerous; we must have one set theoretic system as the foundation for all areas of mathematics.

Nevertheless, many mathematicians see ZF+DC as a sufficient set theory for all forms of constructive, applicable mathematics. However, these mathematicians are in the vast minority: let alone the Hahn-Banach theorem, the Axiom of Choice is firmly entrenched in modern mathematics, and removing it would hinder the development of mathematics. An example of its necessity is its relation to Tychonoff's theorem, which states that the Cartesian product of compact sets is itself compact. This statement, itself equivalent to the Axiom of Choice, is one of the most important theorems of point-set topology; as with the Hahn-Banach theorem, being unable to prove this result would too severely restrict the development of this area of mathematics.

There are further problems with using ZF+DC as our axiom system of set theory. In section 2.1 we discussed how the existence of a finitely additive solution to the problem of measure is solved in the positive for $\mathbb{R}^1$ and $\mathbb{R}^2$, provided we were working in an axiom system of set theory having the Axiom of Choice. Without this, Solovay was able to prove that ZF+DC+GM is consistent if and only if ZF+DC+NM is consistent, where NM is an axiom that states there is no additive measure on all subsets of $\mathbb{R}^n$ such that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^n$. This is, in effect, a negation of the theorem of ZF+AC that proves the existence of such a measure on $\mathbb{R}^1$ and $\mathbb{R}^2$. So if we use ZF+DC as our underlying set theory, then the assumption that the problem of measure is solved in the positive is a consistent theory if and only if the assumption that the problem of measure is solved in the negative is also a consistent theory. We are therefore relying on the possibility of a negative resolution of the problem of measure in order to positively resolve it.

3.3 Verdict

On the whole, it must be concluded that one cannot afford to mathematically resolve the Banach-Tarski paradox by removing the Axiom of Choice from our underlying axioms of set theory. In order to retain complete mathematical consistency, the Axiom of Choice is necessary in all areas of mathematics, as theorems such as the Hahn-Banach theorem and Tychonoff's theorem are too important to remove.

Though this argument is often put forward by mathematicians in favour of retaining the Axiom of Choice, they neglect to note that these two examples only require the axiom for certain cases. In the Hahn-Banach theorem, for example, the Axiom of Choice is only needed if the vector space in question is non-separable. Similarly, Tychonoff's theorem only requires the full strength of the axiom if the product in question is uncountable. For the other cases, these theorems can be proved in ZF+DC. So removing the Axiom of Choice does not remove all mathematical development based on these two theorems. Nevertheless, the generality of these theorems would be lost, and many further theorems rely on the full strength of these two examples.

On the other hand, removing the Axiom of Choice would, at least in measure theory, return the consistency of the properties of space and mathematical models thereof, and the Banach-Tarski paradox could be, in some sense, satisfactorily resolved. It can be argued that most mathematical theorems applicable to modelling the physical world are provable in ZF+DC, and this is certainly the conviction of a significant number of mathematicians. This conviction, however, must be coupled with the acknowledgement that by limiting applicable mathematics to the axiom system of set theory of ZF+DC, applied mathematics and pure mathematics are being firmly divided into two areas, where bridges between the disciplines must be made with care and the understanding that the difference in foundations may lead to inconsistent results.

Chapter 4

Measurability

In the original formulation of the problem of measure, it is asked whether such a measure exists when defined on all subsets of $\mathbb{R}^n$. The existence of non-measurable sets shows the danger of this assumption. By removing these sets from consideration, we greatly simplify the theory of measures. So it seems natural to ask whether the restriction of measure to certain subsets of $\mathbb{R}^n$ results in a positive solution to the problem of measure.

Question 3. *Using the standard axioms of set theory, on what subsets of* $\mathbb{R}^n$ *does there exist a measure that is countably additive, isometry invariant, and gives the unit cube volume 1? Is this measure unique?*

4.1 Non-Measurable Sets

In many early theories of measure, it was necessary to determine whether a set was "measurable" or not; informally, whether a set was "as large from the inside as the outside". The standard way to do this was, for bounded sets, to compare a set's measure to the measure of its complement. This approach was taken by Lebesgue in his groundbreaking theory of measure. Lebesgue was heavily influenced by the work of Borel, under whose measure closed sets were measurable. In fact, Borel showed that under his measure, the σ-algebra of Borel sets (the smallest σ-algebra containing the open sets) consists purely of measurable sets.

In most early versions of measure, the need to specify which sets were measurable was quite necessary, as many simple sets did not have well-defined measures; under Cantor's measure, for example, the set $[0, 1] \cap \mathbb{Q}$ is not measurable. Lebesgue measure is much more consistent than this, in that essentially all "natural" sets are measurable. Nevertheless, Vitali was able, in 1905, to construct a Lebesgue non-measurable set using the Axiom of Choice.

Lebesgue's attitude towards this was circumspect; though his measure defines whether a set is measurable or not, he personally did not believe in Vitali's construction, rejecting its usage of the Axiom of Choice.

The concern that a set may appear large from the outside and small from the inside turns out to be very valid, as the Banach-Tarski paradox most strikingly shows. By manipulating non-measurable pieces of B^3, Banach and Tarski showed that volume could be, in a sense, duplicated. This extreme inconsistency in measure theory highlights the need to only consider volumes of measurable sets.

4.2 The Restriction of Measures to Measurable Sets

Lebesgue's restriction of his measure to the σ-algebra of measurable sets paved the way for a similar approach in the development of abstract measure theory. As seen through the definition of a measure, it is now considered standard to only consider measures over a restricted set of subsets of a space. The choice of a σ-algebra as the restriction is very natural; as the standard operations involving manipulations of sets are complements, unions, and intersections, this means that not only are σ-algebras closed under these operations, but that by generating the smallest σ-algebra containing just a few sets, many useful sets can be constructed. In particular, the most useful sets in $\mathbb{R}^n$ in nearly all applications are those somehow closely related to open or closed sets, and the σ-algebra of Borel sets contains all of these.

The derivation of Lebesgue measure on $\mathbb{R}^n$ is to begin by considering either intervals, if we are working in $\mathbb{R}^1$, or rectangles, if we are working in $\mathbb{R}^n$, and to define their volume naturally as the product of their side lengths. We can then determine the volume for all sets closely related to these rectangles by considering σ-algebras that contain these sets. The smallest such σ-algebra is the Borel σ-algebra. Alternatively, we could consider the slightly larger σ-algebra consisting of all sets that satisfy Carathéodory's criterion of measurability.

The latter is the standard approach taken in constructing Lebesgue measure on $\mathbb{R}^n$. With this, it is possible to prove many major theorems of measure and integration theory without losing too much information. In particular, Lebesgue measure ensures that the measure of the region under the graph of a Riemann integrable function is identical to its Riemann integral, as well as guaranteeing if the region under the graph of a function is Lebesgue non-measurable, then the function must not be Riemann integrable. Lebesgue measure restricted to measurable sets is also countably additive, which helps resolve issues in the development of Fourier series for when, for a sequence of functions $(f_n)_{n=1}^{\infty}$, we have that $\sum_{n=1}^{\infty} \int f_n = \int \sum_{n=1}^{\infty} f_n$.

On the other hand, as it has been the norm for all of the 20th century to consider measures over σ-algebras, it is difficult to say what might have been proven in other definitions of a measure. The focus has been on considering measures restricted to measurable sets, on which a strong field of measure and integration theory has developed. So far, these fields have appeared to be consistent and sufficient for most applications to physical world problems.

The restriction of Lebesgue-measure to non-measurable sets is widely accepted as a satisfactory resolution of the problem of measure; it is the unique isometry invariant measure defined on the Lebesgue-measurable sets that normalises the unit cube. In some sense, it is also a resolution of the Banach-Tarski paradox, explaining that volume cannot be duplicated with measurable sets. Nevertheless, criticism remains of this approach; for one, it limits the sets over which one can define a measure, losing the generality of being able to perform operations in measure theory over all sets. As a resolution of the Banach-Tarski paradox, it does not deny the physical possibility of the duplication of volume.

These criticisms, however, are outweighed in the mathematical community by the support of restrictions of measures. For it must again be noted that non-measurable sets are highly pathological and non-constructive, and so although they might exist heuristically in real life, it could never be known how to actually create them. But most of all, mathematicians claim that by considering measures over only measurable sets, they have unlocked a rich and consistent area of mathematics in measure theory; by restricting to a σ-algebra of sets for which a measure acts over, many argue that much stronger theorems can be proved than under more general conditions. In the development of measure theory to this day, mathematicians have so far found that though outer measures and metric outer measures are more general than measures in acting on all sets, it is not possible to prove nearly as many apparently useful theorems with these systems than with measures.

4.3 Verdict

Restricting Lebesgue measure to the σ-algebra of Lebesgue-measurable sets is a mathematically satisfactory solution to the problem of measure. It is, in some sense, begging the question to suggest that restricted measures lead to a more powerful theory of measure than (outer) measures acting on all sets, as the mathematical focus of the last century has been firmly focussed on the former. In spite of this, it cannot be disputed that the restricted measure approach has been effective in leading to a consistent form of measure theory that is general enough to prove most of its original motivations, such as problems in Riemann integration or Fourier analysis.

But we must still ask whether this approach adequately resolves the disparity between the physical world and mathematics? If we apply the problem of measure to its main motivation in determining a volume structure for space, we have no choice to conclude that it is not possible to define some notion of volume to every region of space. This appears, at face value, to be quite illogical; it seems bizarre that one region of space might have a well-defined volume, whereas another region might not. Thus we must remark that though a mathematically consistent solution to the problem of measure, restricted measures fail to adequately model three-dimensional volume in the physical world.

Finally, the argument put forward by many mathematicians on the pathological nature of non-measurable sets seems to be fallacious. Though they rightly suggest that the Axiom of Choice creates highly pathological and non-constructive non-measurable sets, they nevertheless cannot deny the existence of such a set. But merely the existence of such a set suggests the physical possibility of the Banach-Tarski paradox, which is in direct contradiction of many of the laws of physics, such as the conservation of energy.

We must therefore conclude that the restriction of measure is only an adequate answer to the problem of measure in the realm of pure mathematics. It fails to provide a suitable model of the volume of space in the physical world, and hence fails to offer a resolution of the Banach-Tarski paradox.

Chapter 5

Conclusion

The Banach-Tarski paradox is a mathematical theorem that contradicts man's natural intuition of the conservation of volume: that the sum of the volumes of a (finite) number of pieces is equal to the volume of the union of the pieces. This paradox is also a proof in the negative to answer Lebesgue's problem of measure: using the standard axioms of mathematics, does there exist a measure on all the subsets of $\mathbb{R}^n$ that is countably additive, isometry invariant, and gives the unit cube volume 1? If so, is this measure unique? In this paper, we discussed whether weakening certain conditions of the problem of measure led to a positive solution, and the implications of such a solution on the Banach-Tarski paradox.

For $\mathbb{R}^1$ and $\mathbb{R}^2$, one can simply weaken the condition of additivity from countable to finite, and then the problem of measure is solved by an extension of Lebesgue measure onto all subsets. This resolution is perfectly adequate; the cost is merely the monotone convergence theorem. This weakens slightly the development of integration theory in $\mathbb{R}^1$ and $\mathbb{R}^2$, but it nevertheless remains consistent. Unfortunately, relaxing the additivity condition does little to resolve the Banach-Tarski paradox, which occurs in a higher dimension for which the paradox itself acts as a proof of the non-existence of a finitely additive measure. Moreover, non-additive measures as a model for the physical world notion of volume are merely blatant violations of the conservation of energy. Thus it is infeasible to consider altering the condition of additivity of the problem of measure for dimensions higher than 2.

If we accept the Banach-Tarski paradox as purely a paradox of abstract mathematics, then we are fine to consider the existence of non-measurable sets as an adequate resolution of the issue. Lebesgue measure is the unique measure on the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^n$ that is countably additive, isometry invariant, and normalises the unit cube. Moreover, restricting measures to measurable sets has allowed mathematicians, following in the footsteps of Lebesgue, to resolve many outstanding problems in measure and integration theory. But despite the mathematical consistency of this standard approach in measure theory, non-measurable sets fail to adequately address the physical implications of the Banach-Tarski paradox. The restriction of measure suggests that we cannot give every physical body a well-defined volume. But this does not prevent one from working with non-measurable sets, which can unexpectedly lead to the duplication of volume via the Banach-Tarski paradox. Physically, this duplication violates the laws of physics; new mass, and hence new energy, has been created out of nothing, contradicting the conservation of energy. As a result, we cannot consider the restriction of measure as a satisfactory resolution of the Banach-Tarski paradox for applicable mathematics.

On the other hand, the removal of the Axiom of Choice clearly resolves the Banach-Tarski paradox. For without the Axiom of Choice, we are unable to create the non-measurable sets needed for the paradoxical decomposition of B^3. Though we cannot prove that the problem of measure holds true in ZF+DC, it is consistent with ZF to assume that this is the case. The greatest advantage of dismissing the Axiom of Choice is the reuniting of properties of space and its mathematical models. ZF+DC is, in many respects, a better foundation of mathematics than ZF+AC for mathematical theorems modelling the physical world, not least for its removal of the duplication of volume apparent through the Banach-Tarski paradox. Yet though altering the axioms of set theory is effective in retaining consistency in the physical world, it introduces inconsistency in mathematics itself. The Axiom of Choice is integral to many areas of mathematics, where it has been used to prove foundational theorems upon which vast regions of further study have ensued. Thus one cannot remove the Axiom of Choice entirely from all mathematics. The only remaining possibility is to have one set of axioms – ZF+DC+GM – for applied mathematics, and another – ZF+AC – for pure mathematics. This, of course, leaves mathematics open to inconsistencies when these two areas are bridged.

So we have found two resolutions to the problem of measure: one that retains the consistency of mathematics, and one that retains the consistency of the physical world. For resolving the Banach-Tarski paradox, our primary motivation throughout this paper, removing the Axiom of Choice is clearly the superior approach. Despite this, it remains an inelegant solution to what is, essentially, an unsolvable problem: none of the approaches taken in this paper give an outcome that is positive both mathematically and physically.

Perhaps we must return again to the problem of measure and considering altering other conditions. Perhaps instead of showing the need for volume to only be defined on measurable sets or only being used in a set theory without the Axiom of Choice, the Banach-Tarski paradox is a proof of the inadequacy of $\mathbb{R}^n$ as a model of space. Solid objects, for example, consist of a finite number of atoms. As the pieces used in the paradoxical decomposition of B^3 are uncountable, this implies that no paradoxical decomposition of a solid object can physically occur. Nevertheless, the duplication of space alone is enough to violate properties of fields in physics. Thus the Banach-Tarski paradox is conceivebly evidence that an uncountable model is insufficient. In fact, work of Randall Dougherty and Matthew Foreman shows the existence of Banach-Tarski type decompositions in any complete separable metric space. In order to avoid this, a countable incomplete model of space is needed. Like the other proposed resolutions of the problem of measure, however, this proposal comes at the cost of other properties generally desirable in modelling space.

The Banach-Tarski paradox has no elegant solution, as a true paradox ought to. The standard approach of mathematicians, to dismiss the paradoxical decomposition as being irrelevant due to the non-measurability of the pieces involved, does little to allay the physical concerns raised by the paradox. The most popular alternative involves the resolution of these physical concerns through the alteration of the axioms of set theory, but in doing so introduces disparities between areas of mathematics. As a paradox, Banach and Tarski's contradictory decomposition of a ball may have no graceful resolution, but the beauty of this problem is the development of knowledge ensuing in efforts to understand and explain its issues.

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