Bolzano and the modernisation of mathematical method -boltz'n'all

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Mathematics and Philosophy are two fields of thought that have always been very closely linked. However, throughout the ages, as philosophy has expanded and split into more specialized modes of thought and discovery, so too has the distance between mathematics and philosophy grown. A number of key events throughout history can be identified as contributing to this separation of fields. One such event was the birth of calculus. This discovery by Newton and Liebniz highlighted the immense power of mathematics but also enforced the importance for mathematicians to demand rigour in their mathematical proof and theory. It was Bolzano who sought to restore a rigorous method and good practice to the field of mathematics by reinterpreting and reformulating the Ancient Greek ideals in light of modern mathematical discovery. The influence of Bolzano's philosophies on mathematics remain strong today.

The Mathematics of the Ancient Greeks

To understand how Bolzano's contributions to the field of mathematics, and in particular the rigour of mathematical proof and endeavour, brought it back in line with the original ideals of mathematics, it is important to understand the history of mathematics and how the development of calculus threatened these ideals. While today the fields of mathematics and philosophy are generally viewed as existing quite independently, it can be seen from examining the philosophy of the Greeks, both grew from the desire to discover more about reality. Furthermore, it becomes clear that each discipline had a strong effect on the other. The Ancient Greeks are generally credited with extending primitive philosophy and mathematics to a more developed form by harnessing the unique power of human beings to reason and rationalise. They believed that these skills could be used to discover the underlying truths about the world around them. As such, they questioned the world around them and the principles that governed events. Their intellectual endeavours and theorising proved successful and they started unlocking some of the patterns in the world around them. Encouraged by this success and the perceived power of logical reasoning, the Greeks took one step further. They claimed that the universe is mathematically designed, and through mathematics man can penetrate to that design (Kline, 1982, p11). This thesis was interpreted and implemented differently by the various schools of thought, however by examining all of their interpretations, the strong link that existed between mathematical and philosophical ideas and method becomes clear.

The Pythagoreans took this idea quite literally. They saw numbers not as the abstract concepts that we see them as today, but instead as physical points or particles (Kline, 1982, p12). They believed that number was the essence of all physical things and this was why explanation of the physical world could only be undertaken in the language of number, mathematics. Numerical relationships governed their explanation of the world to the extent that they predicted the presence of ten celestial bodies, and imagined those they could not observe, because ten was considered to be the ideal number (Kline, 1982, p14). For Plato however, numbers and geometry existed in an ideal world that transcended time. He believed that the“knowledge at which geometry aims is knowledge of the eternal, and not of aught perishing and transient” (Kline, 1982, p16). Supposedly, mathematicians were able to catch glimpses of the ideal world through the practice of geometry. In fact, Plato believed that in the quest for knowledge of reality mathematics could be considered a substitute for physical investigation and experiment (Kline, 1982, p17).

Through the formation of the theories presented above, the Greek mathematicians developed a framework of thought that has transcended millennia of changing theories and ideas on the nature of reality. The field of mathematics developed by the Greeks can be identified by three main ideas. Firstly, they required that mathematical concepts be abstract concepts that are permanent, unchanging and therefore applicable to a variety of problems. Secondly, they developed the concept of axioms; truths that were considered self-evident that were to be used as a starting point for all mathematical reasoning(Kleiner, 1991, p293). Finally, they decided that mathematical reasoning should be conducted in a deductive manner and that this would guarantee the truth of the conclusion. This mathematical structure has survived the ages, and while mathematics briefly lost sight of it during the birth of calculus, it was reemphasised by various mathematicians including Bernard Bolzano. By examining the events surrounding the development of the calculus it is possible to observe how these ideals were compromised.

The concepts of calculus

The ideas behind our notion of calculus, in particular the problem of the infinite, have been around since the time of the Greek mathematicians. The Greeks believed that the infinite was linked with chaos and therefore the opposite of perfection. Despite this dislike of the infinite they could not avoid it as is was ingrained in various observables “in time, in the generations of men, and in the division of magnitudes” (Barnes et al., 1984, Phys. III 6, 206a18). As Aristotle observed, without the concept of the infinite “there will be a beginning and an end of time, a magnitude will not be divisible into magnitudes, number will not be infinite" (Barnes et al., 1984, 206a10-12). Thus the Greeks could not escape the idea of infinity and so instead they defined the concept rather ambiguously. According to Aristotle "the alternative then remains that the infinite has a potential existence There will not be an actual infinity" (Kline, 1982, p199). This idea that we cannot talk about anything ever reaching or achieving infinity but rather that we must talk about the potential to do so was deemed a satisfactory solution to the problem of infinity for thousands of years, despite it being far from a permanent, well defined concept.

Mathematicians were confronted with the issue of infinity and the need to clarify and strengthen their understanding of it with the development of the calculus in the form that we recognise it today. In the 17th century Liebniz and Newton demonstrated the immense power and broad possibilities of the calculus. Unfortunately, it relied on the unfamiliar and ill-defined concept of an infinitely small number. While historically mathematicians had been concerned with infinitely big quantities, the infinitely small quantities of calculus were challenging in a similar way. Namely, their presence raised the question of what happens when a process is repeated unendingly, an idea that today we may recognise as a limit. While they managed to solve many problems that had been outstanding since the time of the Ancient Greeks, even Liebniz and Newton themselves, struggled to justify and define these infinitely small numbers, how they behaved and how they could be used. Liebniz later defended himself from "over-precise" critics stating that excessive scrupulousness should not be let to get in the way of mathematical endeavour and invention (Kline, 1982, p137). It was views like these that threatened the Ancient Greeks' rigorous basis of mathematics.

The power and use of calculus was widely accepted and appreciated, however after years of operating with undefined concepts of the infinitely small, mathematicians were becoming more worried about the strength of the grounding of calculus. Calculus was threatening the foundations of mathematics as it depended upon concepts that were not well defined or understood and furthermore, mathematical reasoning was being misappropriated in an attempt to justify these concepts. It was in this new age, where mathematicians were calling for the foundations of calculus to be strengthened, that Bolzano began his mathematical career.

Bolzano's Influences

During his time as an undergraduate at university Bolzano studied both Philosophy and Mathematics, however in his later life Philosophy became his primary focus. This led Bolzano to examine mathematical problems from a different perspective, that of a philosopher. As Bolzano wrote

> My special pleasure in mathematics rested therefore particularly on

> its purely speculative parts, in other words I prized only that part of mathematics which was at the same time philosophy.(O'Connor and Robertson, 2005)

Given the atmosphere in the field of mathematics at the time and Bolzano's philosophical interests, he found himself addressing questions concerning mathematical rigour and procedure, and how they should be achieved. He was looking for answers to questions such as: How do we define a mathematical object? What makes a rigorous proof? What is the relationship between the various areas of mathematics? Perhaps most importantly he approached these questions with the aim of strengthening past results and not just those to come. In particular he questioned those results that mathematicians of the time took for granted. As Bolzano wrote

> I decided to observe the following rule: no obviousness of an assumption will force me to neglect the duty to seek its proof till I see clearly that no proof can be required and why it cannot be required.

(Folta, 1981, p19) . While Bolzano was disapproving of the tendency of many of the mathematicians of the time to accept results and theories as obvious he was also unimpressed by the calibre of the proofs that were constructed. He believed that mathematics could only act as a tool to exercise correct thinking if it was presented and conducted appropriately (Folta, 1981, p19) . Often the faults he found in the presentation of mathematical theories, proof and thinking related to a misinterpretation or misapplication of the Greek ideals.

Bolzano's beliefs on Mathematical Rigour

Bolzano's ideas of mathematical rigour, and how it was to be achieved, centred around the idea that mathematics consisted of different areas or levels of concepts. He made the distinction between specific areas of mathematics, for example geometry, compared with general or pure areas of mathematics, such as analysis. According to Bolzano, specific areas of mathematics could only ever provide good applications or examples of the more general ideas and theories of the pure mathematics. Concepts that existed in the area of pure mathematics were abstract concepts that needed to be well defined before they could be used. This reflected the ideals of the ancient Greeks regarding what is to be considered an appropriate mathematical object.

This ideal was challenged in the development of calculus with the idea of infinity. As outlined above, calculus relied heavily on the notion of an infinitely small number. However, mathematicians of the time were not able able to define what such a number was without calling on geometric or intuitive proofs. Bolzano did not just highlight the need for a more abstract and concrete definition of the concept, he demonstrated how that could be done by helping to develop the 'epsilon-delta' definition of the concept of the limit as a number becomes increasingly smaller. Bolzano defined the existence of a limit of a function in the following way:

> for all real $\epsilon > 0$ there exists a real $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, we have$|f(x) - L| < \epsilon$

(Exner, 1999, p2). Note that in mathematics ϵ and δ are generally used to signify a small number. From this definition we can see that the limit is clearly defined by providing bounds on its possible location that become more precise as we alter the input. This definition not only strengthened the foundations of calculus but also provided a standard for mathematical definitions that reemphasised the need for abstract, precise concepts.

More importantly Bolzano also called for the reexamination of proofs, and in particular the criteria to determine which axioms and arguments were appropriate to use. This was in response to the growing number of mathematicians relying on geometric or other specific proofs to demonstrate and justify the broader, analytic, concepts behind calculus. Bolzano saw these proofs as examples not proofs. He agreed that examples and applications dont generally detract from the integrity of scientific endeavour if provided on that pretext. However, he insisted that by offering such examples as proofs of pure concepts the stability and rigour of mathematics was being greatly compromised (Russ, 2004, p256). Instead Bolzano demanded that general mathematical proofs be constructed from axioms appropriate to the area and furthermore that deductive reasoning be used in the place of reasoning specific to any particular area.

This stance reflects a similar attitude held by many of the Greek mathematicians. As can be seen from the following summary of a story from Plutarchs "Life of Marcellus", Plato had had a very similar opinion, "Eudoxes and Archymtas used physical arguments to "prove" mathematical results Plato indignantly denounced such proofs as a corruption of geometry; they utilised sensuous facts in place of pure reasoning" (Kline, 1982, p17). The influence of the Greek ideals in the modern work around calculus was evident in that Newton "admitted much later that he used analysis to find the theorems in the Principles but he formulated the proofs geometrically to make the arguments as secure as the ancients"(Kline, 1982, p135). This was concerning as it showed that the philosophy of the Ancient Greeks was being misinterpreted and poorly applied in the modern mathematical climate.

The modernisation of ancient ideals

What this does highlight is that there was a need for these ideals to be reformulated in light of modern mathematical discovery. More specifically, Bolzano clarified that while Plato saw the purest form of mathematics as being geometry, in this new mathematical climate, geometry was only a specific area of mathematics and instead analytic, algebraic or other principles were required to construct general proofs. Bolzano wrote,

> ... it is also equally clear that it is an unacceptable breach of good method to try to derive proof of pure (or general) mathematics (ie. arithmetic, analysis, algebra) from considerations which belong to a merely applied (or special) part of it, namely geometry.

(Russ, 2004, p254) So Bolzano did more than simply reiterate the ideals presented by the Ancient Greeks, he encouraged mathematicians to accept the underlying philosophies of the Greeks while applying them in a manner more appropriate to modern mathematics.

While Bolzano's contributions to the field of mathematics are perhaps best understood by examining the development of calculus, they in fact transcend particular areas or disciplines to provide a solid structure for the study of mathematics. The mathematical method he proposed, and the rigour he demanded of mathematicians, provided guidance and stability in a very uncertain period. Bolzano recognised that while the Greek mathematicians provided a solid framework for mathematical endeavour, their message needed to be reformulated to suit a modern mathematical climate. Moreover, his work provides a precedent for future mathematicians and philosophers for the need to reinterpret mathematical method and philosophy in light of major mathematical advances.

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