THE PHILOSOPHY OF FRACTALS

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Abstract. In this report I will take a brief look at fractals from a philosophical point of view. In particular the definition of a fractal, it's computability, it's relationship with the axiom of choice and whether fractals really exist in nature.

1. Introduction

Fractals are mathematical objects which most people know look very fascinating especially when produced under particular algorithms. Our facination with fractals seems to have something to do with the fact that we see many fractal-like objects in nature. But do fractals really exist in nature and in fact do they 'exist' at all. Whilst we can describe them with a few equations, are they computable and can we ever really see a whole fractal? Are we assuming the axiom of choice when we produce a fractal using the chaos game and if the axiom of choice is not true could we produce a fractal? These are some of the questions that I'll ponder in this paper. We start with the very definition of a fractal.

2. What is a Fractal?

It seems that different texts tend to have a different definition of what a fractal is. Part of this is due to the extensive variety of fractals, many looking quite different with very different properties. This makes it very hard to find the things they have in common could be used to define them. Benoit Mandelbrot gave a definition in his book *The Fractal Geometry of Nature* then later withdrew it since it wasn't really suitable for some fractals. The simplest and most common definition of a fractal is the following [5]:

Definition 1. *A fractal is a set that consists of smaller subsets similar to the larger set in some way.*

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This is a very broad definition and allows any possibility determining the similarity of sets. Some examples include being a scaled down versoin, having the same Hausdorff dimension or perhaps even just having to contain a line and a point. This definition also doesn't specify that the smaller subsets should also consist of smaller subsets similar to the whole subset. i.e. a fractal should have self-similarity evident on all scales. For example take a (solid) disk in the plane, one could argue that it contains smaller disks inside and that therefore the larger disk is a fractal. One should note however that this definition can still imply self-similarity on all scales, for example take an image of three equilateral triangles with each touching a vertex of the other two. If you then said each triangle is a shrunken image of all three then each triangle must contain three smaller ones. However if you look at this result then each of these triangles is not a shrunken image of the whole so we change each of the three to shrunken versions of the whole and repeat this forever (refer to figure1). This is one construction of the Sierpinski triangle.

creation.jpg

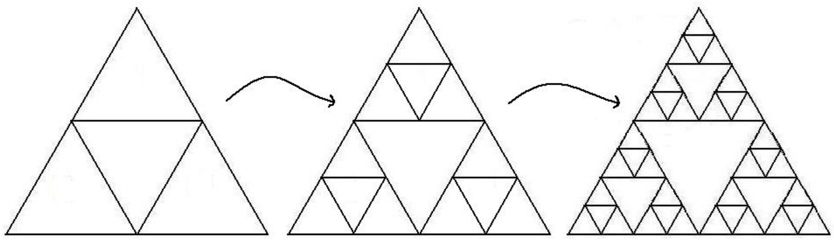


FIGURE 1. creating the Sierpinski triangle

Mandelbrot gave an alternative definition of a fractal but to understand this I must first explain the notion of fractal dimension.

2.1. **Fractal Dimension.** Fractal dimension is a measure of the size and complexity of a fractal. It can be calculated precisely by the box counting algorithm. This is a consequence of the box counting theorem which states [Super Fractals p.175, Barnsley]
Let $\mathcal{A} \in H(\mathbb{R}^m)$, where the Euclidean metric is used. Cover $\mathbb{R}^m$ by closed square boxes of side length $(1/2)^n$. Let $N_n(A)$ denote the number of boxes of side length $(1/2)^n$ which intersect the attractor.
If $D = \lim_{n\to\infty} \frac{\ln(N_n(\mathcal{A}))}{\ln(2^n)}$ exists then D is defined as the fractal dimension and $\mathcal{A}$ has fractal dimension D.

This is a very precise number but can be experimentally estimated by evaluating $\frac{\ln(N_n(\mathcal{A}))}{\ln(2^n)}$ for finite n. For example, given a digital image the smallest size box we can use to measure is a single pixel. If an image had resolution 1024 × 1024 then we could do at most 11 steps with box lengths (in pixels) given by 2^{10-n} where $n = 0, 1, 2, ..., 9, 10$. You should realise this is a very bad way of finding fractal dimensions and is fundamentally wrong. Fractal dimension can only really be found on objects with infinite resolution since we must take the limit of n to infinity.
The Hausdorff-Besicovitch fractal dimension (abbreviated Hausdorff dimension) is defined in a much more analytical way but is always no more than the fractal dimension. In fact we often find they are the same which is why it can also be referred to as the fractal dimension. It is defined as follows [2]:

Definition 2. *Let $S \subset \mathbb{X}, \delta > 0$ and $0 \leq s < \infty$. Let*
$H^S_\delta(S) = \inf\{\sum_{i=1}^{\infty} \|U_i\|^s : \{U_i\}$ *is a δ-cover of $S\}$,*
Then the s-dimensional Hausdorrf measure of S is $H^S(S) = \lim_{\delta\to 0} H^S_\delta(S)$.
The Hausdorff dimension of $S \subset \mathbb{X}$ is defined to be $dim_H S = \inf\left\{s : H^S(S) = 0\right\}$.

Now we can make sense of Mandelbrot's alternative definition of a fractal [5]:

Definition 3. *A fractal is a set whose Hausdorff dimension is larger than it's topological dimension.*

For example, all arcs have a topological dimension of 1. An arc traces out a fractal if it's Hausdorff dimension is more than 1. For example the seirpinski triangle (whose limit set is a curve) has Hausdorff dimension $\log(3)/\log(2) \approx 1.585 > 1$ and is hence a fractal. The Sierpinski triangle also satisfies the previous definition since it can be broken up into 3 shrunken versions of itself, each of which can be again broken up and so on.

3. Fractals and the axiom of choice

Most fractals we see can be defined mathematically without the axiom of choice but do our methods of computing a fractal implicitly use the axiom of choice? In particular we need to know if the chaos game requires the axiom of choice. If this was true the computability of fractals would depend on whether the axiom of choice held.

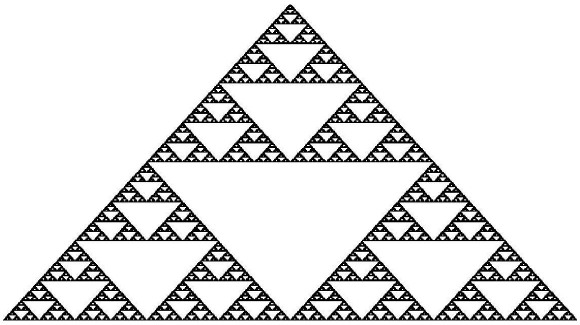
triangle.jpg 

FIGURE 2. Sierpinski triangle

Definition 4 (IFS (iterated function system)). *An IFS is a collection of functions that map a subset of a metric space into itself. It is denoted by* $F = \{\mathbb{X}; f_1, ..., f_n\}$.

For example the Sierpinki triangle lies in the Euclidean plane and consists of 3 functions, hence $F = \{\mathbb{R}^2; f_1, f_2, f_3\}$. We generally require the functions to be contractive so that a fractal can produced from the IFS in which case we say the IFS is hyperbolic. I will look at fractals produced by a hyperbolic IFS in this essay. A fractal is produced from an IFS in the following way:
The Chaos game is where we choose a function from those in the IFS (iterated function system) at random according to predetermined probabilities and then plot the point given by the function. We then take that point and repeat the process and by doing this enough times we get an approximate image of our fractal. To completely generate a fractal we'd have to pick our first point to be on the fractal and then play the chaos game for an infinite amount of time. For example, take the IFS for the sierpinski triangle and take the origin. If we apply $f_2, f_3, f_2, f_1, f_3, ...$ in a very long random sequence and plot each point as we go then we would produce a set looking a lot like the Sierpinski triangle and if the sequence were infinite we would produce the Sierpinski triangle.
Another way of generating a fractal is to take an image and apply all of the IFS functions to it, superimpose the resulting images then repeat with the new image. Do we require the axiom of choice to do this? The fractal produced by an IFS is called the attractor and is denoted $\mathcal{A}_{\mathcal{F}}$.
We know that the axiom of choice doesn't apply if we can enumerate the set. One could enumerate the set according to the order in which they came out using the chaos game and for points that come uptwice we just keep the order to which we first assigned it. This seems a little

vague and it will also produce quite a mess in the sense that the ordering has nothing to do with where the point is on the fractal. Also, the enumeration will be differnent each time you play the chaos game and there's also the fact that you need to play it forever to get every point. As a result we don't really get an enumeration.

Definition 5 (code space [4])**.** *A code space* Ω *is a space of addresses* $\sigma = \sigma_1, \sigma_2, ...$ *Suppose* $\phi : \Omega \to \mathbb{X}$ *then if* $\phi(\sigma) = x$ *we call* σ *the address of* x*.*

Another possible way to order a fractal is in code space. This is an extremely useful space and it is through this space that we can construct fractal homeomorphisms. If a fractal is generated by a hyperbolic IFS which consists of a set and n functions, $\mathcal{F} = \{\mathbb{X}, f_1, ..., f_n\}$, then we construct a code space Ω whose elements are an infinite sequence of numbers from the set $\{1, ..., n\}$. For example, let $\sigma \in \Omega$ then σ has the form $\sigma = \sigma_1, \sigma_2, ..., \sigma_i, ...$ where each $\sigma_i \in \{1, ..., n\}$.
These elements correspond to points on a fractal as follows. For example we know that if we apply a fixed infinite series of the functions $f_{\sigma_1} \circ f_{\sigma_2} \circ ... \circ f_{\sigma_i} \circ ...$, then the result will correspond to a single point on the fractal. i.e.
$y = \lim_{k \longrightarrow \infty} f_{\sigma_1} \circ f_{\sigma_2} \circ ... \circ f_{\sigma_k}(x)$ for all $x \in \mathbb{X}$. Hence we assign $y \in \mathcal{A}$ (a point on the fractal) the code $y \to \sigma = \sigma_1\sigma_2...\sigma_i....$ We can then define a function $\phi_{\mathcal{F}}$ which maps elements of the code space to points on a fractal, $y = \phi_{\mathcal{F}}(\sigma) = lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ ... \circ f_{\sigma_k}(x)$, where $x \in \mathbb{X}$. This mapping from code space to the fractal attractor has a nice geometrical interpretation. Each fuction maps a point to a specific part of the attractor. I won't go into too much detail but consider the example of the sierpinski triangle. There are three subtriangle similar to the whole by a scaling of 1/2. f_1, f_2, f_3 will correspond to one of the three parts each. Inside each of these smaller triangles we see the same thing. By going down further and further we obtain a finite sequence of functions corresponding to a subtriangle. (see figure3)

There is however a slight problem that more than one code may map to the same point. For example, on the sierpinski triangle we can see that $1\bar{2} = 2\bar{1}$. To fix this we restrict ourselves to the tops code space which has a one to one correspondence with points on the fractal. Then using this codespace we can define an ordering relation on the fractal set.
Suppose we have σ, ω which correspond to the same point on a fractal i.e. $\phi_{\mathcal{F}}(\sigma) = \phi_{\mathcal{F}}(\omega)$. Let k be the first index for which the codes σ and ω don't match up. Hence $\sigma_i = \omega_i$ for all $i < k$ and $\sigma_k \neq \omega_k$, and we then define the ordering relation $\sigma < \omega$ iff $\sigma_k > \omega_k$. First you should

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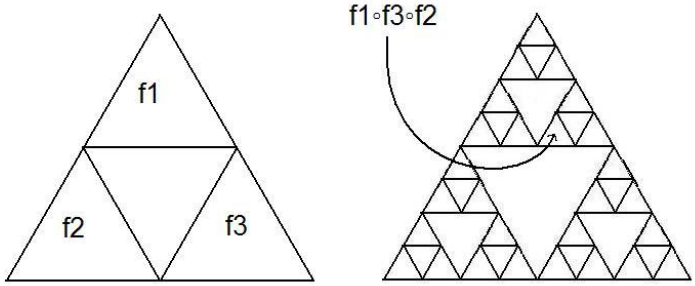


FIGURE 3. functions mapping into the Sierpinski triangle

note that this ordering relation seems kind of backwards, for example note that all σ are greater or equal to $\bar{n} = nnnnnnn...$ and less than or equal to $\bar{1} = 1111111....$ This relation is will be very useful to us.
Now with the ordering relation above we define $\tau_{\mathcal{F}}(x) = \max\{\sigma \in \Omega : \phi_{\mathcal{F}}(\sigma) = x\}$. Then the tops codespace is defined as $\Omega_{\mathcal{F}} := \{\tau_{\mathcal{F}}(x) : x \in \mathcal{A}_{\mathcal{F}}\}$. $\Omega_{\mathcal{F}}$ and $\mathcal{A}_{\mathcal{F}}$ have a one to one correspondence and hence an ordering of $\Omega_{\mathcal{F}}$ is equivalent to an ordering of the fractal $\mathcal{A}_{\mathcal{F}}$. Hence we can use the ordering relation above to define an ordering over $\Omega_{\mathcal{F}}$. This however is not an enumeration since given two distinct codes in $\Omega_{\mathcal{F}}$ it may always be possible to find another in between. This makes it as difficult as trying to enumerate the real line segment $[0, 1]$. However what I'll now discuss will show you that we want to know if this is a well-ordering.

3.1. **Computability of fractals.** Some fractals can be quite difficult to describe and what does this have to say about the computability of a fractal. The problem is that fractals are sets of infinitely many elements and to 'compute' a fractal we might say we need to be able to list it's elements in some sort of way. This brings us to the well-ordering principle which states any set E can be well-ordered which is defined as follows [Real analysis p.26]:

Definition 6. *A set E is linearly ordered if there is a binary relation $\leq$ such that:*

(a) *$x \leq x$ for all $x \in E$*

(b) *If $x, y \in E$ are distinct, then either $x \leq y$ or $y \leq x$ (but not both)*

(c) *If $x \leq y$ and $y \leq z$ then $x \leq z$*

Definition 7. *A set E can be well-ordered if it can be linearly ordered in such a way that every non-empty subset $A \subset E$ has a smallest element in that ordering.*

It is well known that the well-ordering principle is equivelent to the axiom of choice, i.e. each implies the other.

3.1.1. *Ordering of a fractal set.* Now given an IFS I showed above we can get a one-to-one correspondance with a fractal and it's tops code space and that we can then apply an ordering to the tops code space. It is clear that using this ordering the set $\Omega_{\mathcal{F}}$ is linearly ordered. However to know if it is well-ordered we need to know if every non-empty subset of $\Omega_{\mathcal{F}}$ has a smallest element. First consider the whole of $\Omega_{\mathcal{F}}$. I mentioned that all $\sigma \in \Omega_{\mathcal{F}}$ are greater or equal to $\bar{n} = nnnnnnn...$ according to our relation. Hence you might be tempted to say that $\bar{n}$ is the least element of $\Omega_{\mathcal{F}}$, however it is not clear that $\bar{n} \in \Omega_{\mathcal{F}}$. For example consider $\sigma \in \Omega$ such that $\sigma \neq \bar{n}$ and $\phi_{\mathcal{F}}(\sigma) = \phi_{\mathcal{F}}(\bar{n}) = y$. If such a σ doesn't exist then there is no problem but if one does then we would find that $\bar{n} \notin \Omega_{\mathcal{F}}$ since $\tau_{\mathcal{F}}(y) = \sigma$ since $\bar{n} < \sigma$. As a result it becomes unclear if $\Omega_{\mathcal{F}}$ has a least element.
To avoid this problem we can define a new ordering relation after the tops code space has been produced. This will essentially be a reversed version of the ordering we've bee using i.e. $\sigma < \omega$ iff $\sigma_k < \omega_k$ where k is the least index for which $\sigma_k \neq \omega_k$. Then $\bar{1}$ must be in $\Omega_{\mathcal{F}}$ as a result of the ordering used to construct $\Omega_{\mathcal{F}}$ whilst this new ordering guarantees this is now the least element. This gives hope that $\Omega_{\mathcal{F}}$ can be well ordered but what we really need to know is any subset $A \subset \Omega_{\mathcal{F}}$ has a least element and if $\bar{1} \notin \mathcal{A}$ then this is again unclear. Whist there must be an infimum there is no guarantee that this lies in $\mathcal{A}$. Consider $B = \Omega_{\mathcal{F}}/\{\bar{1}\}$, the inf of B is clearly $\bar{1}$ but what is the least element? So it seems we may either assume it can be well-ordered or assume that it can't, and of course assuming the first means we must assume the well-ordering principle is true. This means we are also assuming the axiom of choice is true, so by trying to avoid the axiom of choice we are led to it's equivalent.
This doesn't imply the axiom of choice and the well ordering pricipal are necessary and it also doen't imply that they aren't necessary. However it does seem at this point that if you wished to enumerate or well-order every fractal set it could only be done if they were both true.

4. Is a coastline really a fractal?

This isn't easy to answer? They certainly look like they might correspond to a fractal, like a Julia set perhaps.

We could approximate the Hausdorff dimension by tracing the coastline from a map and applying the box-counting algorithm (see the section on notions of dimension). If we then refer to definition 2 we would find that most coastlines have a Hausdorff dimension more than 1 (which is the topological dimension) and could therefore be considered to be fractals. However if we refer to definiftion 1 we may see some self-similarity in a coastline however I must point out that this self-similarity doesn't continue indefinitely on all scales as it does in most mathematical fractals. I would therefore argue that a coastline is not a fractal in this sense. Hence we appear to have contradicting arguements.

One important question is just how did we compute the Hausdorff dimension of the coastline. This is most generally approximated by the box-counting algorithm in which case we must ask at what scale this was done to. If we took a map of a particular coastline and zoomed in far enough we would see that it consists of a finite piecewise smooth curve. This then means the coastline in fact has Hausdorff dimension 1. Hence we could then say that a coastline is definitely not a fractal according to our definitions.

However you may then ask, how accurate was the map we used, what if we had a map outlining the coastline accurately on all scales. Then we need some sort of precise definition of the coastline, an obvious definition would be the line which separates land from water. However this is constantly changing with the waves and the tide. To solve this we could gather data over some time and then take the mean or alternatively just look at an instant in time. Ultimately we would get a coastline accurate to the scale of millimeters but this could still be broken into a smooth piecewise curve. Eventually we find that the smallest scale we could measure the coastline down to is were water molecules meet sand molecules. Again we could describe it as a piecewise smooth curve. Even if we went further into particle physics and quantum physics we would be stopped by the Planck length assuming our current understanding of physics is correct.

So is a coastline a fractal? No, not strictly according to our definitions. However it would be possible to create a fractal which, when smoothed at the appropriate length scale, would be exaclty our coastline. Hence we may view coastlines as an approximation to a fractal. Such a fractal could be produced by the collage theorem. In simple terms this theorem states that if you cover an image with smaller copies of itself and then work out the functions that map the image to each of the smaller pieces then you can create an IFS with these functions and then produce a fractal from the IFS which will look as close as you like to the original image. For example examine a coastline to the length scale at

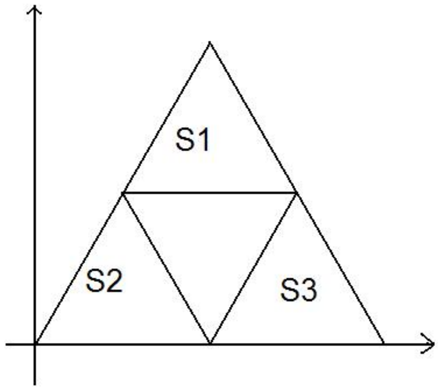
which we can't get any more accurate simply place a shrunken segment of the original coastline and continue this on all scales. As a result we would have a fractal that looks like the coastline at regular scales but has fractal properties at all scales. Also there are many algorithms to create coastline looking curves using fractal and chaos theory. This is why many people may consider coastlines to be fractals.
The same could be applied to ferns, trees, landscapes, rivers and any fractal-like object in nature. They're all fractals in the sense they can easily be approximated by fractals but strictly speaking they aren't. Of course all of this assumes that our current physics and understanding of the universe is correct which brings me (very briefly) onto the topic of fractal cosmology .

4.1. **Fractal cosmology.** Not only do people see examples of fractals in nature but also in the universe. Some have suggested that galaxies are fractally distributed and the notion of fractal hasn't only been used on coastlines but also on some of the many things out in space. There is also the question of whether or not the universe is bigger than the observable universe. This applies to infinite and infinitesimal scales. For example the universe may have infinite resolution going beyond the Planck length but we just can't observe it. Given the probabilistic nature of quantum mechanics it doesn't seem surprising that there are chaotic systems out there that may produce fractal looking objects or have many other fractal like properties. Who knows what we might find in the future.

5. Can Fractals be described over alternatives to ZF?

This is another very interesting question since if we could produce a set theory where we don't have the complications of the axiom of choice then it could make many things quite a lot simpler and perhaps we could even produce a more concrete definition of a fractal. Using definition 1 it may be very easy to produce a fractal using other axioms of set theory. What we really need to know is which axioms of ZF are necessary to define a fractal. I will look at a few of these.
The first of the axioms of ZF, which is also in most of the alternatives to ZF in some form, is the notion of equality of sets. i.e. that two sets are equal if and only if they contain exactly the same elements. This is extremely important in all of set theory but is especially relevant to fractals since this will be neccesary to show the self-similarity of sets. For example if we look at the Sierpinski triangle, S, and separate it into 3 parts, S_1, S_2 and S_3 we can see that $2 \times S_2 = S$ (by which we mean $\{2x : x \in S_2\} = \{x : x \in S\}$) and similarly for S_1 and S_3 with appropriate translations. But using the equality sign only makes sense

if we have this axiom of ZF.



3 subsets.jpg

FIGURE 4. similar subsets of the Sierpinski triangle

The third axiom is the axiom schema of specification and would be necessary to define some fractals and is also very usefull in theory. For example we can't define the tops codespace without this axiom since we require to separate the max of all elements in codespace that correspond to the same point of a fractal. This axiom allows us to do this.

The axiom of infinity is obviously necessary in ZF due to the infinite nature of fractals. For a specific example consider the subset of a fractal produced by taking a single element x and adding the element $f_1(x)$ to the subset and then $f_1(f_1(x))$ etc. Such a set only exists under the axiom of infinity. However whilst this axiom is necessary to produce infinite sets in ZF there are alternatives like NF which have infinite sets without this axiom.

We could go on to find that most of the axioms of ZF are necessary and that some may be better replaced with others but we'd essentially have the same thing. I won't go into this because of the amount of detail necessary. You should note however that it is unclear whether the axiom of choice and the well-ordering principle should be part of ZF due to some of the paradoxes they bring up. However many people include them in what is abbreviated ZFC, i.e. ZF with the axiom of choice, in order to avoid problems. As I discussed above it is unclear whether we need it but it may be easier to include it.

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